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Engineering Notes

Lunar Synchronous Orbits in the Earth–Moon Circular-Restricted Three-Body Problem

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Introduction

THE circular-restricted three-body problem for the Earth–moon system gives a simplified description of the motion of a spacecraft under the gravitational influence of the Earth and the moon, but it is useful for future space missions that involve the stationing of a spacecraft near a Lagrangian point or in a lunar polar orbit [1]. In fact, a communication satellite at the L_1 point provides continuous communication coverage between the Earth and the lunar surface. A communication satellite in a halo orbit near the L_2 point serves as a relay satellite for communications with the far side of the moon [2]. Relay satellites in a lunar polar orbit could also provide a communication network between the Earth and the far side of the moon.

It is known that there are many types of periodic solutions for the circular-restricted three-body problem [3–7]. Because of the potential use for future lunar missions, libration orbits are mainly investigated. From the point of view of lunar south pole coverage, L_1 and L_2 halo families with a period between 7 and 16 days are investigated in detail [8]. However, periodic orbits of the circular-restricted three-body problem that are synchronous with the motion of the Earth–moon two-body problem are outside of these families. Lunar synchronous orbits fall in a class of cycler orbits that transfer a spacecraft from one body back to the same body [9–11], where an additional requirement to visit some specified points such as Lagrangian points can be imposed. For lunar synchronous orbits, their period is specified rather than pointing to visits on the way.

In this Note, generation and maintenance of lunar synchronous orbits are discussed. First, a coplanar orbit that lies in the orbit plane of the Earth–moon system will be generated by iteratively adjusting initial conditions. Rotating the coplanar orbit about an axis of the barycenter inertial frame and adjusting initial conditions, a noncoplanar orbit is also generated. Considering the linearized system around a periodic orbit, eigenvalues of the monodromy matrix are calculated. To maintain these orbits, feedback controls are designed using the linear quadratic regulator theory of periodic systems [12]. For a given perturbation of initial conditions numerical simulations are performed, and the total velocity change and the settling times are calculated. The former turns out to be very small.

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Circular-Restricted Three-Body Problem

Consider the circular-restricted three-body problem for the Earth–moon–spacecraft system [1,3]. Here, equations of motion of the spacecraft will be described by two coordinate systems (X, Y, Z) and (x, y, z), where the former is an inertial frame whose origin is the barycenter of the Earth–moon system, and the X-Y plane is the orbit plane of the circular motion, while the latter is a rotating reference frame (synodic frame) that rotates with the Earth–moon system about the z axis. Initially, the Earth and the moon are assumed to be on the X axis and the two coordinate systems are assumed to be identical. Thus, the Earth and the moon are fixed on the x axis, and the system (x, y, z) coincides with the inertial system periodically. The equation of motion of the spacecraft follows from Newton's second law and is given by

$$\ddot{\mathbf{R}} = -GM_e \frac{\mathbf{R} - \mathbf{R}_e}{|\mathbf{R} - \mathbf{R}_e|^3} - GM_m \frac{\mathbf{R} - \mathbf{R}_m}{|\mathbf{R} - \mathbf{R}_m|^3} \tag{1}$$

where $\mathbf{R}$, $\mathbf{R}_e$, and $\mathbf{R}_m$ are the position vectors relative to the barycenter, of the spacecraft, the Earth, and the moon, respectively, G is the universal gravitational constant, and M_e and M_m are masses of the Earth and the moon, respectively. In the (X, Y, Z) system, Eq. (1) in nondimensional form becomes

$$\ddot{X} = -\frac{(1-\rho)(X + \rho\cos t)}{R_1^3} - \frac{\rho[X - (1-\rho)\cos t]}{R_2^3}$$
$$\ddot{Y} = -\frac{(1-\rho)(Y + \rho\sin t)}{R_1^3} - \frac{\rho[Y - (1-\rho)\sin t]}{R_2^3}$$
$$\ddot{Z} = -\frac{(1-\rho)Z}{R_1^3} - \frac{\rho Z}{R_2^3} \tag{2}$$

where

$$R_1 = (1-\rho)[(X + \rho\cos t)^2 + (Y + \rho\sin t)^2 + Z^2]^{1/2}$$
$$R_2 = \rho[(X - (1-\rho)\cos t)^2 + (Y - (1-\rho)\sin t)^2 + Z^2]^{1/2}$$

The mass ratio is $\rho = M_m/(M_e + M_m)$, and time is in units of $1/n \approx 4.3$ [day], where n is the angular velocity of the rotating frame, while distance is in units of $D \approx 380,000$ [km], the distance between the Earth and the moon. The period of the Earth–moon system is $T_{\mathrm{syn}} = 2\pi/n$.

In the synodic frame, the equations of motion in nondimensional form are given by

$$\ddot{x} - 2\dot{y} - x = -\frac{(1-\rho)(x+\rho)}{r_1^3} - \frac{\rho(x - 1 + \rho)}{r_2^3}$$
$$\ddot{y} + 2\dot{x} - y = -\frac{(1-\rho)y}{r_1^3} - \frac{\rho y}{r_2^3}, \qquad \ddot{z} = -\frac{(1-\rho)z}{r_1^3} - \frac{\rho z}{r_2^3} \tag{3}$$

where $r_1 = [(x+\rho)^2 + y^2 + z^2]^{1/2}$ and $r_2 = [(x - 1 + \rho)^2 + y^2 + z^2]^{1/2}$. The state space form of Eq. (3) is

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ x + 2\dot{y} - \frac{(1-\rho)(x+\rho)}{r_1^3} - \frac{\rho(x-1+\rho)}{r_2^3} \\ y - 2\dot{x} - \frac{(1-\rho)y}{r_1^3} - \frac{\rho y}{r_2^3} \\ -\frac{(1-\rho)z}{r_1^3} - \frac{\rho z}{r_2^3} \end{pmatrix} \equiv \mathbf{f}(\mathbf{x}) \tag{4}$$

where $\mathbf{x} = (x \quad y \quad z \quad \dot{x} \quad \dot{y} \quad \dot{z})^T$. In the next section, initial conditions of Eqs. (2) and (3), which give lunar synchronous orbits (periodic solutions with period $T = 2\pi$), will be numerically generated.

Lunar Synchronous Orbits

To find lunar synchronous orbits, Eqs. (2) and (3) are solved numerically for a given set of initial conditions, and numerical values after one period are compared with the initial values. Updating initial conditions, numerical calculations are repeated until the terminal values coincide with the initial conditions. First a coplanar synchronous orbit that is symmetric with respect to the X axis is sought, so that initial conditions $Z(0) = 0$, $\dot{Z}(0) = 0$, $\dot{X}(0) = 0$, and $Y(0) = 0$ are assumed. Then $z(0) = 0$, $\dot{z}(0) = 0$, $\dot{x}(0) = 0$, and $y(0) = 0$. Initial approximations of $X(0)$ and $\dot{Y}(0)$ are derived from those at perigee of Keplerian orbits with period 2π, ignoring the effects of the moon, because the mass ratio of the Earth and the moon is small ($\rho = 0.01215$). Note that $x(0) = X(0)$, and the relative velocity $\dot{y}(0)$ is calculated by $\dot{y}(0) = \dot{Y}(0) - x(0)$.

To find initial approximations, choose an eccentricity e and determine $X_0(0)$ of the elliptic orbit as follows. Recall that the period of the Earth–moon system follows from Kepler's third law and is given by

$$T_{\mathrm{syn}} = 2\pi\sqrt{\frac{D^3}{G(M_e + M_m)}}$$

The semimajor axis of an elliptic orbit of the Earth–spacecraft system with period T_{syn} is then given by

$$a = \sqrt[3]{GM_e\frac{T_{\mathrm{syn}}^2}{2\pi}} = \sqrt[3]{1-\rho}D$$

Thus, the nondimensional semimajor axis is $\bar{a} = \sqrt[3]{1-\rho}$. Because the nondimensional perigee distance is $\bar{r}_p = \bar{a}(1-e)$ and the X coordinate of the Earth is $-\rho$, one obtains

$$X_0(0) = x_0(0) = \sqrt[3]{1-\rho}(1-e) - \rho$$

The nondimensional perigee velocity $\bar{v}_p$ of the Earth–spacecraft system is

$$\bar{v}_p = \sqrt{\frac{2}{\bar{r}_p} - \frac{1}{\bar{a}}} = \sqrt[3]{1-\rho}\sqrt{\frac{1+e}{1-e}}$$

Because the velocity of the circular motion of the Earth around the barycenter is ρ, $\dot{Y}_0(0)$ is given by

$$\dot{Y}_0(0) = \sqrt[3]{1-\rho}\sqrt{\frac{1+e}{1-e}} - \rho \tag{5}$$

and the velocity in the synodic frame is given by

$$\dot{y}_0(0) = \dot{Y}(0) - x(0) = \sqrt[3]{1-\rho}\sqrt{\frac{1+e}{1-e}} - \sqrt[3]{1-\rho}(1-e)$$

To obtain proper initial conditions of Eq. (2), $e = 0.8$ and 0.9 are considered. Then the initial condition $\dot{Y}_0(0)$ of the elliptic orbit is calculated by Eq. (5). The initial condition $\dot{Y}(0)$ of Eq. (2) is modified as $\dot{Y}_0(0) + \delta v$, $X(0) = X_0(0)$, and the error $(X(2\pi) - X(0), Y(2\pi) - Y(0))$ with parameter δv is plotted. For $e = 0.8$, δv in the interval $[4.18 \times 10^{-3}, 4.37 \times 10^{-3}]$ is considered. $X(2\pi) - X(0)$ is always negative, but $Y(2\pi) - Y(0) = 0$ is realized. On the other hand, for $e = 0.9$ and $\delta v \in [1.51 \times 10^{-3}, 1.61 \times 10^{-3}]$, there is a subinterval where $X(2\pi) - X(0)$ is positive and $Y(2\pi) - Y(0) = 0$ is realized. Thus, there is an $e \in (0.8, 0.9)$ for which the periodic conditions are satisfied. We follow a standard bisector method, and set $e_2 = (e_0 + e_1)/2$ with $e_0 = 0.8$ and $e_1 = 0.9$ and examine the error as above. Repeating this process, $e = 0.835054$ and $\delta v = 3.16919 \times 10^{-3}$, which give a 2π periodic orbit, are obtained. Initial conditions thus obtained are

$$\begin{aligned}&(X(0), Y(0), Z(0), \dot{X}(0), \dot{Y}(0), \dot{Z}(0))\\&\quad = (0.152125, 0, 0, 0, 3.31290, 0)\\&(x(0), y(0), z(0), \dot{x}(0), \dot{y}(0), \dot{z}(0))\\&\quad = (0.152125, 0, 0, 0, 3.16077, 0)\end{aligned}$$

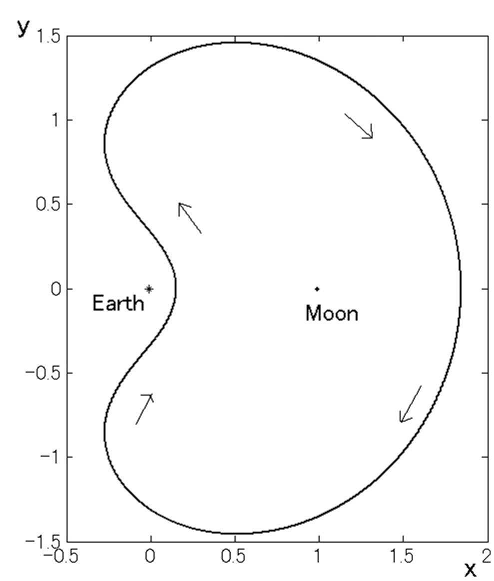

Fig. 2 Coplanar orbit in the synodic frame.

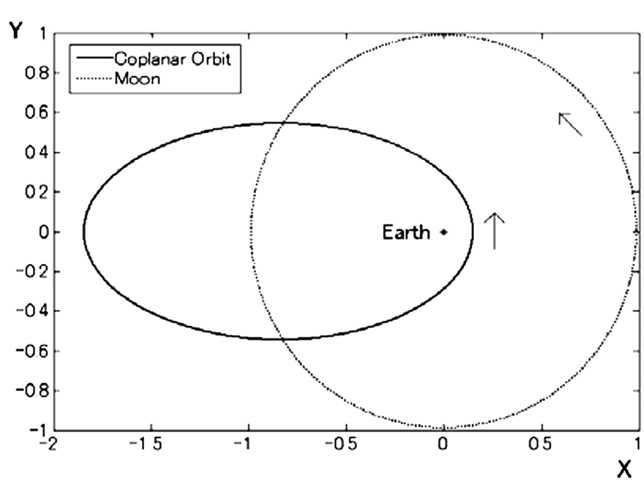

Fig. 1 Coplanar orbit in the inertial frame.

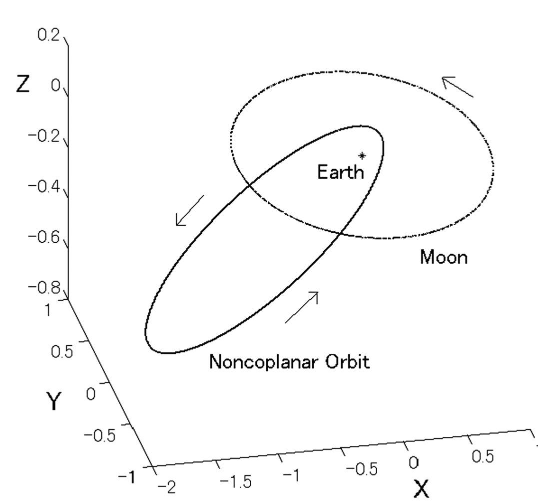

Fig. 3 Noncoplanar orbit in the inertial frame.

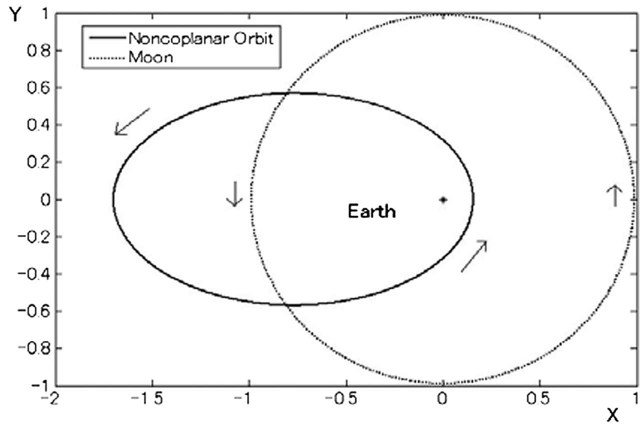

Fig. 4 Projection on *X*-*Y* plane.

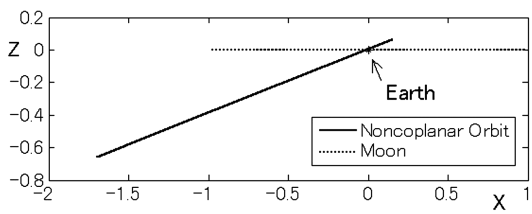

Fig. 5 Projection on *X*-*Z* plane.

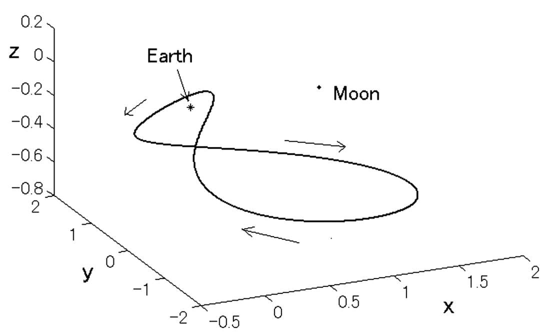

Fig. 6 Noncoplanar orbit in the synodic frame.

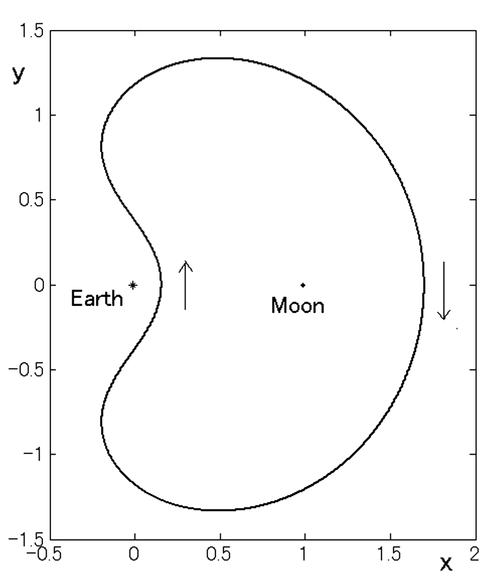

Fig. 7 Projection on *x*-*y* plane.

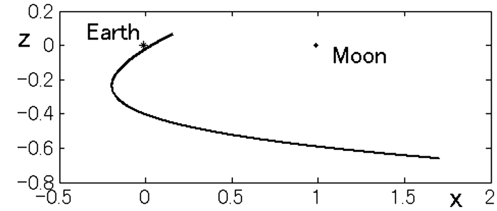

Fig. 8 Projection on *x*-*z* plane.

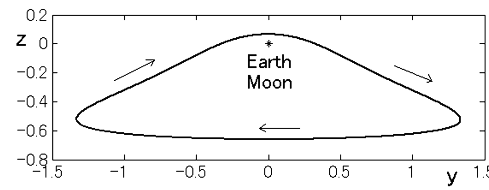

Fig. 9 Projection on *y*-*z* plane.

The orbit in the inertial frame is given in Fig. 1 and it is almost elliptical, while in the synodic frame it is bean shaped (see Fig. 2). The closest point to the Earth is approximately 6.325×10^4 km. The Jacobi's integral for Eq. (3) is -1.0443 and differs from those of coplanar orbits of the Earth–moon system obtained by Prado and Broucke [10].

To find a synchronous orbit with nonzero $Z(0)$, the initial condition of the coplanar orbit is rotated about the Y axis. Then the initial condition becomes

$$\mathbf{X}(0) = \left(\sqrt[3]{1-\rho}(1-e)\cos i - \rho, 0, \sqrt[3]{1-\rho}(1-e)\sin i, 0, \sqrt[3]{1-\rho}\sqrt{\frac{1+e}{1-e}} - \rho + \delta v, 0\right)$$

where i is an inclination angle. In the synodic frame

$$\mathbf{x}(0) = \left(\sqrt[3]{1-\rho}(1-e)\cos i - \rho, 0, \sqrt[3]{1-\rho}(1-e) \times \sin i, 0, \sqrt[3]{1-\rho}\sqrt{\frac{1+e}{1-e}} - \sqrt[3]{1-\rho}(1-e) + \delta v, 0\right)$$

The angle i is increased with increment 0.01 and the errors $X(2\pi) - X(0)$, $Y(2\pi) - Y(0)$, and $Z(2\pi) - Z(0)$ are computed. For each i, $X(2\pi) - X(0) = 0$, $Y(2\pi) - Y(0) = 0$ are realized by adjusting two parameters e and δv. The error $Z(2\pi) - Z(0)$ is positive and increases initially with i, but starts decreasing. The error $Z(2\pi) - Z(0)$ becomes zero when $i = 0.370037$, $e = 0.818519$, and $\delta v = 3.34225 \times 10^{-3}$. Initial conditions obtained in this way are as follows:

$$(X(0), Y(0), Z(0), \dot{X}(0), \dot{Y}(0), \dot{Z}(0)) = (0.156358, 0, 0.0653656, 0, 3.14383, 0)$$

$$(x(0), y(0), z(0), \dot{x}(0), \dot{y}(0), \dot{z}(0)) = (0.156358, 0, 0.0653656, 0, 2.98747, 0)$$

The lunar synchronous orbit in the inertial frame is given in Fig. 3. It is almost elliptical, but not planar. Projections of the orbit on *X*-*Y* and *X*-*Z* planes are given, respectively, in Figs. 4 and 5. The orbit in the synodic frame (the relative orbit) is depicted in Fig. 6 , which looks like a bent ellipse. Projections of the orbit on three planes are given in Figs. 7–9.

No noncoplanar orbits by rotation about the X axis are obtained.

Stability of the Synchronous Orbits

In this section, stability of the lunar synchronous orbits is examined. Let $\mathbf{x}_p$ be a synchronous solution of Eq. (4). Then the linearized equation of Eq. (4) at $\mathbf{x}_p$ is given by

$$\dot{\mathbf{e}} = \frac{\partial \mathbf{f}(\mathbf{x}_p)}{\partial \mathbf{x}} \mathbf{e} \equiv A(t)\mathbf{e} \tag{6}$$

where $\mathbf{e} = \mathbf{x} - \mathbf{x}_p$ is the deviation from the synchronous solution, and $\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}$ is the Jacobian given by

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial z} & 0 & 2 & 0 \\ \frac{\partial f_5}{\partial x} & \frac{\partial f_5}{\partial y} & \frac{\partial f_5}{\partial z} & -2 & 0 & 0 \\ \frac{\partial f_6}{\partial x} & \frac{\partial f_6}{\partial y} & \frac{\partial f_6}{\partial z} & 0 & 0 & 0 \end{pmatrix}$$

where

$$\frac{\partial f_4}{\partial x} = -\frac{(1-\rho)[r_1^2 - 3(x+\rho)^2]}{r_1^5} - \frac{\rho[r_2^2 - 3(x-1+\rho)^2]}{r_2^5} + 1$$

$$\frac{\partial f_5}{\partial y} = -\frac{(1-\rho)(r_1^2 - 3y^2)}{r_1^5} - \frac{\rho(r_2^2 - 3y^2)}{r_2^5} + 1$$

$$\frac{\partial f_6}{\partial z} = -\frac{(1-\rho)(r_1^2 - 3z^2)}{r_1^5} - \frac{\rho(r_2^2 - 3y^2)}{r_2^5}$$

$$\frac{\partial f_4}{\partial y} = \frac{\partial f_5}{\partial x} = \frac{3(1-\rho)(x+\rho)y}{r_1^5} + \frac{3\rho(x-1+\rho)y}{r_2^5}$$

$$\frac{\partial f_4}{\partial z} = \frac{\partial f_6}{\partial x} = \frac{3(1-\rho)(x+\rho)z}{r_1^5} + \frac{3\rho(x-1+\rho)z}{r_2^5}$$

$$\frac{\partial f_5}{\partial z} = \frac{\partial f_6}{\partial y} = \frac{3(1-\rho)yz}{r_1^5} + \frac{3\rho yz}{r_2^5}$$

This is a periodic system, and to examine its stability eigenvalues of the monodromy matrix $\Phi(2\pi, 0)$ are calculated, where $\Phi(t, s)$ is the transition matrix of Eq. (6) that transfers the state at s to the current state at t. Monodromy matrices of coplanar and noncoplanar orbits are given, respectively, by

$$\Phi_{\mathrm{cp}}(2\pi, 0) = \begin{pmatrix} -0.0013 & -0.0000 & 0 & 0.0000 & -0.0001 & 0 \\ -0.2412 & 0.0012 & 0 & 0.0001 & -0.0201 & 0 \\ 0 & 0 & 0.0001 & 0 & 0 & -0.0000 \\ 2.4093 & -0.0108 & 0 & -0.0010 & 0.2009 & 0 \\ 0.0157 & 0.0002 & 0 & 0.0000 & 0.0014 & 0 \\ 0 & 0 & -0.0000 & 0 & 0 & 0.0001 \end{pmatrix} \times 10^4$$

and

$$\Phi_{\mathrm{ncp}}(2\pi, 0) = \begin{pmatrix} -0.0010 & -0.0000 & -0.0004 & -0.0000 & -0.0001 & -0.0000 \\ -0.1778 & 0.0010 & -0.0621 & 0.0001 & -0.0179 & 0.0000 \\ -0.0004 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 \\ 1.4070 & -0.0073 & 0.4917 & -0.0007 & 0.1419 & -0.0003 \\ 0.0113 & 0.0002 & 0.0039 & 0.0000 & 0.0012 & 0.0000 \\ 0.6160 & -0.0032 & 0.2154 & -0.0004 & 0.0621 & -0.0000 \end{pmatrix} \times 10^4$$

The set of their eigenvalues denoted by σ_{cp} and σ_{ncp} are given, respectively, by

$$\sigma_{\mathrm{cp}} = \{0.470629 \pm 0.882331i, 0.999996 \pm 0.002758i, 0.979039, 1.021410\},$$

$$\sigma_{\mathrm{ncp}} = \{0.493929 \pm 0.869502i, 0.999622 \pm 0.027494i, 0.999335, 1.000665\}$$

Four complex eigenvalues are on the unit circle, one real eigenvalue is inside the unit disk, and one is outside. However, the magnitudes of the unstable eigenvalues are small. The stability index [8] is $\nu_{\mathrm{cp}} = 1.000224$ for the former and $\nu_{\mathrm{ncp}} = 1.000000$ for the latter.

Orbit Maintenance

Assuming a slight deviation of initial conditions from those of lunar synchronous orbits, the total velocity change required to bring the spacecraft back to the orbit will be calculated. For a spacecraft with three independent thrusters, Eq. (6) is modified as

$$\dot{\mathbf{e}} = A(t)\mathbf{e} + B\mathbf{u}$$

where $B = [0_{3\times3}, I_{3\times3}]^T$. As for controller design, linear quadratic regulator theory is employed, where the cost function is given by

$$J(\mathbf{u}; \mathbf{e}(0)) = \int_0^\infty (\mathbf{e}^T Q\mathbf{e} + \mathbf{u}^T R\mathbf{u})\, \mathrm{d}t$$

where $Q \geq 0$ and $R > 0$. Because $(A(t), B)$ is controllable, the following Riccati differential equation

$$-\dot{P}(t) = A^T(t)P(t) + P(t)A(t) - P(t)BR^{-1}B^T P(t) + Q \tag{7}$$

has a nonnegative 2π periodic stabilizing solution for any Q such that $(\sqrt{Q}, A(t))$ is detectable [12]. Recall that

$$A(t) = \frac{\partial \mathbf{f}(\mathbf{x}_p(t))}{\partial \mathbf{x}}$$

The initial condition of the 2π periodic solution is calculated recursively, jointly solving Eqs. (4) and (7). In the case of the coplanar orbit

$$P(0) = \begin{pmatrix} 9.0322 & 1.0013 & 0 & 0.1015 & 0.7436 & 0 \\ 1.0013 & 0.3186 & 0 & 0.0321 & 0.0781 & 0 \\ 0 & 0 & 0.2843 & 0 & 0 & -0.0036 \\ 0.1015 & 0.0321 & 0 & 0.0033 & 0.0079 & 0 \\ 0.7436 & 0.0781 & 0 & 0.0079 & 0.0614 & 0 \\ 0 & 0 & -0.0036 & 0 & 0 & 0.0002 \end{pmatrix} \times 10^3$$

while for the noncoplanar orbit

Table 1 Initial values and performance indices: coplanar case

Initial conditions	
Reference orbit	Coplanar orbit
——	$e_{x0} = -1.4 \times 10^{-5}$
——	$e_{y0} = 7.7 \times 10^{-4}$
Initial error	$e_{z0} = -6.4 \times 10^{-7}$
——	$\dot{e}_{x0} = -7.0 \times 10^{-3}$
——	$\dot{e}_{y0} = 1.3 \times 10^{-4}$
——	$\dot{e}_{z0} = 7.1 \times 10^{-4}$
Input weight	$r = 31.6228$
Simulation results	
Velocity change	$\|\mathbf{u}\|_1 = 4.23893 \times 10^{-4}$
Settling time	$T_s = 18.2740$

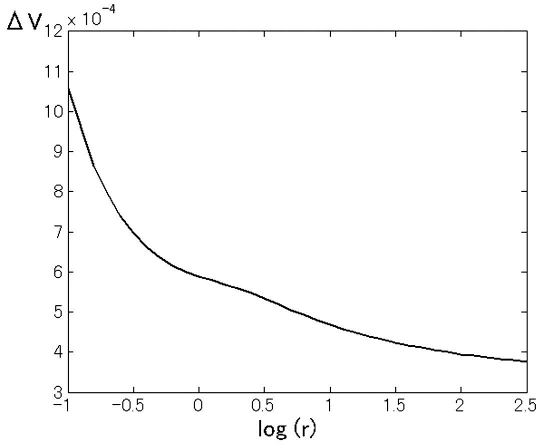

Fig. 10 ΔV vs r.

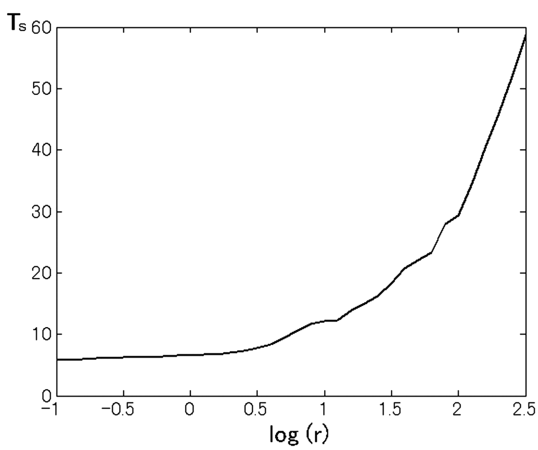

Fig. 11 Settling time vs r.

$$P(0) = \begin{pmatrix} 1.5629 & 0.1393 & 0.5295 & 0.0151 & 0.1563 & 0.0062 \\ 0.1393 & 0.0426 & 0.0504 & 0.0046 & 0.0135 & 0.0018 \\ 0.5295 & 0.0504 & 0.2251 & 0.0056 & 0.0544 & 0.0018 \\ 0.0151 & 0.0046 & 0.0056 & 0.0005 & 0.0015 & 0.0002 \\ 0.1563 & 0.0135 & 0.0544 & 0.0015 & 0.0157 & 0.0006 \\ 0.0062 & 0.0018 & 0.0018 & 0.0002 & 0.0006 & 0.0001 \end{pmatrix} \times 10^4$$

The optimal solution is given by the feedback law $\mathbf{u} = -R^{-1}B^T P(t)\mathbf{e}$. The feedback control is applied to the nonlinear system of the error $\mathbf{e}$ given by

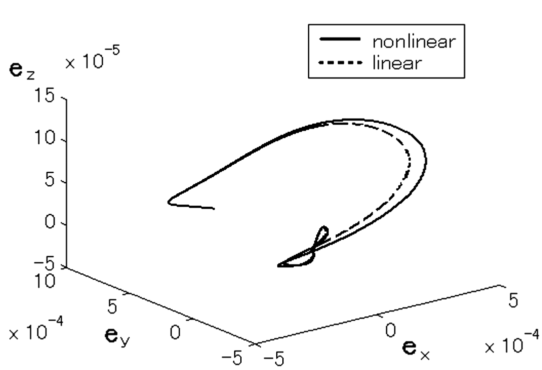

Fig. 12 Convergence to coplanar orbit.

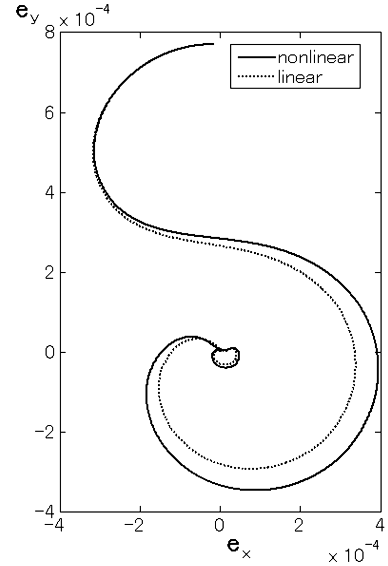

Fig. 13 Projection on x-y plane.

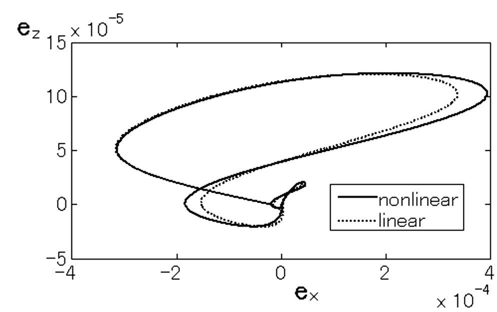

Fig. 14 Projection on x-z plane.

$$\dot{\mathbf{e}} = \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_p) - BR^{-1}B^T P(t)\mathbf{e} \tag{8}$$

For simulations, initial conditions are selected from a neighboring trajectory and given in Table 1. The error $e_{y0} = 7.7 \times 10^{-4}$ in the y direction is largest in the position errors and is equal to 300 km, and the largest velocity error $\dot{e}_{x0}$ is equal to -7.2×10^{-3} km/s. Weight matrices are set to $Q = I$ and $R = rI$. The settling time is defined as the first time after which $[e_x^2(t) + e_y^2(t) + e_z^2(t)]^{1/2} \leq 10^{-3}[e_x^2(0) + e_y^2(0) + e_z^2(0)]^{1/2}$. The total velocity change (L_1 norm of $\mathbf{u}$) and the settling time T_s against the parameter r are given in Figs. 10 and 11, respectively. In view of this, the value of the parameter r is set to $r = 31.6228$.

The controlled trajectory is depicted in Fig. 12, and its projections on the x-y and x-z planes are given in Figs. 13 and 14. The total velocity change is very small and is equal to 4.23893×10^{-4}, which corresponds to 4.34103×10^{-4} km/s. The settling time is

Table 2 Initial values and performance indices: noncoplanar case

Initial conditions	
Reference orbit	Noncoplanar orbit
——	$e_{x0} = -2.0 \times 10^{-5}$
——	$e_{y0} = 9.9 \times 10^{-4}$
Initial error	$e_{z0} = -8.5 \times 10^{-6}$
——	$\dot{e}_{x0} = -7.1 \times 10^{-3}$
——	$\dot{e}_{y0} = 1.8 \times 10^{-4}$
——	$\dot{e}_{ez} = -2.7 \times 10^{-3}$
Input weight	$r = 31.6228$
Simulation results	
Velocity change	$\|\mathbf{u}\|_1 = 5.93896 \times 10^{-4}$
Settling time	$T_s = 23.5781$

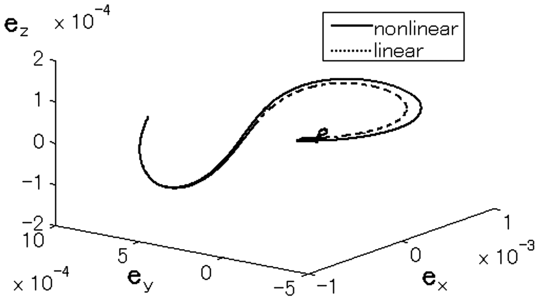

Fig. 15 Convergence to noncoplanar orbit.

$T_s = 18.2740$ (79.4616 days), and it can be made shorter if a smaller r is chosen.

In the noncoplanar case, initial conditions are given in Table 2. The controlled trajectory is depicted in Fig. 15. The total velocity change increases by 40%, and is equal to 6.08207×10^{-4} km/s, and the settling time increases by 30% and is equal to 102.5257 days.

Conclusions

Two lunar synchronous orbits, coplanar and noncoplanar, of the circular-restricted three-body problem for the Earth–moon system are generated by a simple iteration. Eigenvalues of the monodromy matrices are then calculated, which show that both orbits are unstable. However, the stability index is small in either case. For the maintenance of these orbits, feedback controllers from the linear quadratic regulator theory of periodic systems are designed. The total velocity change required is small, as expected.

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References

[1] Wie, B., "Circular Restricted Three-Body Problem," *Space Vehicle Dynamics and Control*, AIAA, Reston, Virginia, 1998, pp. 240–248.

[2] Farquhar, R. W., "Lunar Communications with Libration-Point Satellite," *Journal of Spacecraft and Rockets*, Vol. 4, No. 10, 1967, pp. 1383–1384.
doi:10.2514/3.29095

[3] Szebehely, V., "Description of the Restricted Problem," *Theory of Orbits*, Academic Press, New York, 1967, pp. 7–22.

[4] Zagoura, C., and Kazantzis, P., "Three-Dimensional Periodic Oscillations Generating from Plane Periodic Ones Around the Collinear Lagrangian Points," *Astrophysics and Space Science*, Vol. 61, No. 2, April 1979, pp. 389–409.
doi:10.1007/BF00640540

[5] Breakwell, J., and Brown, J., "The Halo Family of Three-Dimensional Periodic Orbits in the Earth–Moon Restricted 3-Body Problem," *Celestial Mechanics*, Vol. 20, No. 4, 1979, pp. 389–404.
doi:10.1007/BF01230405

[6] Robin, I., and Markellos, V., "Numerical Determinations of Three-Dimensional Periodic Orbits Generated from Vertical Self-Resonant Satellite Orbits," *Celestial Mechanics*, Vol. 21, No. 4, May 1980, pp. 395–434.
doi:10.1007/BF01231276

[7] Howell, K. C., and Breakwell, J., "Almost Rectilinear Halo Orbits," *Celestial Mechanics*, Vol. 32, No. 1, Jan. 1984, pp. 29–52.
doi:10.1007/BF01358402

[8] Grebow, D. J., Ozimek, M. T., Howell, K. C., and Folta, D. C., "Multibody Orbit Architectures for Lunar South Pole Coverage," *Journal of Spacecraft and Rockets*, Vol. 45, No. 2, March–April 2008, pp. 344–358.
doi:10.2514/1.28738

[9] Broucke, R. A., "Traveling Between the Lagrange Points and the Moon," *Journal of Guidance, Control, and Dynamics*, Vol. 2, No. 4, 1979, pp. 257–263.
doi:10.2514/3.55872

[10] Prado, A. F. B. A., and Broucke, R. A., "Transfer Orbits in Restricted Problem," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 3, May–June 1995, pp. 593–598.
doi:10.2514/3.21428

[11] Byrnes, D. V., Longuski, J. M., and Aldrin, B., "Cycler Orbit Between Earth and Mars," *Journal of Spacecraft and Rockets*, Vol. 30, No. 3, May–June 1993, pp. 334–336.
doi:10.2514/3.25519

[12] Ichikawa, A., and Katayama, H., "Continuous-Time Systems," *Linear Time Varing Systems and Sampled-Data Systems*, Springer, London, 2001, pp. 19–27.